POLARIZATION STRUCTURE OF QUANTUM LIGHT FIELDS: A NEW INSIGHT. 2: GENERALIZED COHERENT STATES, SQUEEZING AND GEOMETRIC PHASES

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Abstract

Within the new description of the polarization structure of quantum light (given in Part I) some types of generalized coherent states related to the polarization $SU(2)_p$ group are examined. With their help we give a quasiclassical description of polarization properties of light fields and discuss the concept of squeezing and uncertainty relations for multimode light in the polarization quantum optics. As a consequence, a new classification of polarization states of quantum light is obtained. We also derive geometric phases acquired by different quantum light beams transmitted through "polarization rotators".

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1 Introduction

Recently a new formalism[1-3] was proposed for a description of polarization structure of multimode quantum light fields using the polarization $SU(2)_p$ symmetry and a related concept of the P-quasispin which generalizes the Stokes vector notion at the quantum level and is closely related to the Stokes operators defined in [4]. This approach enabled us to gain a new insight into the polarization structure of light and quantum mechanisms of its depolarization (see Part I (ref. [2]) and references therein).

At the same time, so-called squeezed states of light are intensively examined now within quantum optics (see, e.g., [5-10] and references therein) since these states have attractive properties of the "noise reduction" in measurements of some quantum mechanical observables. However, we note that squeezed states have been studied sufficiently only for single-mode fields [5-7] whereas for multimode fields it is not the case since even the definition of the concept of multimode squeezing is not unique due to a variety of the choices of measurable quantities and appropriate uncertainty measures for them[9-11].

It is well known [11] that the generalized coherent states (GCS)

$$|\{\zeta_i\};\psi_0\rangle = D(g(\{\zeta_i\}))|\psi_0\rangle, \tag{1.1}$$

generated by the action of the displacement operators $D(g(\{\zeta_i\})) = \exp(\sum_i \zeta_i F_i)$ of the groups G^{DS} on certain fixed reference vectors $|\psi_0\rangle$ in the given spaces L^D of the representations $D = \{D(g), g \in G^{DS}\}$ of the groups G^{DS}, present an effective tool for the study of quantum systems having the dynamic symmetry groups G^{DS}. In particular, the average values $\langle\{\alpha_i\};\psi_0|f(\{F_i\})|\{\alpha_i\};\psi_0\rangle$ of the arbitrary functions $f(\{F_i\})$, corresponding to the observables and depending on the generators F_i of G^{DS}, as well as the quasiprobability distribution functions (Q-functions)

$$Q(\{\alpha_i\};\psi_0;\rho) = \langle\{\alpha_i\};\psi_0|\rho|\{\alpha_i\};\psi_0\rangle, \tag{1.2}$$

with ρ being the density matrix, are widely used for vizualizing squeezed states and for a description of quasiclassical properties of the appropriate quantum systems near the "classical limit" [3,6,10-14]. For example, in quantum optics similar quantities, defined using the familiar Glauber's CS and associated with Weyl-Heisenberg group $W(m)$, are widely used for the description of m-mode electromagnetic fields [6,11,13]. The GCS associated with $SU(m)$ groups play the same role for the systems of n-level emitters of radiation [11,15]. Furthermore, the GCS formalism appeared to be useful to calculate geometric phases of quantum systems evolving in time on manifolds with a nontrivial topology [16-19].

The aim of this paper is to examine different kinds of GCS associated with the $SU(2)_p$ group and to apply them for a quasiclassical description of polarization properties of quantum light beams, an analysis of the concept of squeezing of the multimode light related to polarization degrees of freedom and for calculations of specific geometric phases acquired by quantum light beams transmitted through "polarization rotators". In Section 2, for the sake of self-consistency of the exposition, some main points of Part I and related papers are recapitulated. In Section 3 we define and study some sets of the $SU(2)_p$ GCS and determine the Q-representations of different polarization operators providing a quasiclassical description of polarization in quantum optics. In Section 4 the problem of squeezing in polarization quantum optics is discussed using the abovementioned GCS. In particular, we show that quantum states of light beams generated by specific

unpolarized biphoton clusters of the X-type (see Part I) exhibit, in a sense, an absolute squeezing in polarization degrees of freedom. As a consequence, a new classification of polarization states of light within quantum optics is obtained. In Section 5 we calculate "polarization" geometric phases acquired by quantum light beams in different pure states after their transmission through "polarization rotators". In Section 6 some prospects of further developments and applications of results obtained are briefly discussed.

2 Preliminaries

As it was shown in [1,2], in polarization quantum optics there are specific observables which characterize proper polarization properties of light beams and correspond to the group $U(2)$ of a specific polarization gauge invariance of light fields. The generators of this group $U(2)$ in the helicity ($\pm$) polarization basis are of the form

$$P_0 = \frac{1}{2}\sum_{i=1}^{m}[a_+^+(i)a_+(i) - a_-^+(i)a_-(i)] = \sum_i P_0(i), \quad P_\pm = \sum_{i=1}^{m} a_\pm^+((i)a_\pm(i) = \sum_i P_\pm(i),$$

$$2P_2 = i(P_+ - P_-), \quad 2P_1 = (P_+ + P_-), \quad N = \sum_{i=1}^{m}\sum_{\alpha=+,-} a_\alpha^+(i)a_\alpha(i) = \sum_i N(i) \tag{2.1}$$

where m is the number of spatiotemporal (ST) modes under study, N is the total photon number operator and operators P_α are generators of the $SU(2)_p$ subgroup defining the polarization (P) (quasi)spin. The operators P_β and N satisfy commutation relations

$$[N, P_\alpha] = 0, \quad [P_0, P_\pm] = \pm P_\pm, \quad [P_+, P_-] = 2P_0 \tag{2.2}$$

and in the case $m = 1$ coincide up to the factor 1/2 with Stokes operators $\Sigma_\alpha : \Sigma_1 = 2P_2, \Sigma_2 = -2P_0, \Sigma_3 = -2P_1$ [4]. As is clear from Eqs. (2.1) the total P-quasispin components of the field is the sum of the appropriate quasispin quantities for single ST modes. However, from the experimental viewpoint the total P-quasispin enables us to examine new interesting physical phenomena connected with correlations of different modes, in particular, "entangled states" which are widely discussed in multiparticle interferometry [20-22]. Components P_α of the P-quasispin are parameterized on the so-called Poincaré sphere S_P^2 in the classical statistical optics [3,19] and are measurable in polarization experiments related to counting photons with definite circular (P_0) or linear ($P_i, i = 1, 2$) polarizations in quantum optics [23].

Quantities $< P_\alpha >, < N >$ determine the polarization degree $degP$ of light beams with arbitrary wave fronts and frequencies by the relation [2,10]

$$degP = 2[\sum_{\alpha=0,1,2}(< P_\alpha >)^2]^{1/2}/ < N > \tag{2.3}$$

generalizing the appropriate definition [4,13] for the case of a single ST mode. At the same time the quantum averages $< |P^2| >= \bar{p}(\bar{p} + 1)$ of the $SU(2)_p$ Casimir operator $P^2 = (1/2)(P_+P_- + P_-P_+) + P_0^2$ are connected by the relations

$$a) E_r^2 = P^2 - P_0(P_0 - 1), \tag{2.4a}$$

$$b) <|P^2|>=\bar{p}(\bar{p}+1)=\sum_{\alpha=0,1,2}[\sigma^P_\alpha+(<|P_\alpha|>)^2]=\sum_{\alpha=0,1,2}\sigma^P_\alpha+[degP<N>/2]^2 \qquad (2.4b)$$

with the so-called "radial" operator $E_r=\sqrt{P_+P_-}$ used for examining phase properties of electromagnetic fields [24] and with the variances

$$\sigma^P_\alpha=<|P^2_\alpha|>-(<|P_\alpha|>)^2 \qquad (2.4c)$$

determining different uncertainty measures for operators P_α ("polarization noises") [2,10-12].

The $2m$-mode Fock space

$$L_F(2m)=Span\{|\{n_\pm(i)\}>=\prod_{j=1}^{m}(a^+_+(j))^{n^+_j}(a^+_-(j))^{n^-_j}|0\rangle\}$$

is decomposed in a direct sum

$$L_F(2m)=\sum_{p,\sigma}L^{(p,\sigma)}=\sum_{p,n,\lambda}L(p,n,\lambda) \qquad (2.5)$$

of the $SU(2)_p$-invariant subspaces $L(p,n,\lambda)$ which are specified by eigenvalues $p,n,\lambda=[\lambda_i]$ of the P-spin, N and a set of operators describing non-polarization degrees of freedom and are spanned by basis vectors $|p\mu;n,\lambda>$ which are eigenvectors of the operators P^2,P_0,N [25]:

$$P^2|p\mu;n,\lambda>=p(p+1)|p\mu;n,\lambda>,\quad P_0|p\mu;n,\lambda>=\mu|p\mu;n,\lambda>,$$

$$N\mid p\mu;n,\lambda>=n\mid p\mu;n,\lambda> \qquad (2.6)$$

The vectors $|p,\mu;n,\lambda>$ may be expressed in the form of polynomials in operators $a^+_\pm(i),Y^+_{ij},X^+_{ij}$ acting on the vacuum vector$|0>$ where operators

$$Y^+_{ij}=\frac{1}{2}(a^+_+(i)a^+_-(j)+a^+_-(i)a^+_+(j)),X^+_{ij}=a^+_+(i)a^+_-(j)-a^+_-(i)a^+_+(j) \qquad (2.7)$$

are solutions of the operator equations

$$[P_0,Y^+_{ij}]=0;\quad [P_\alpha,X^+_{ij}]=0,\quad \alpha=0,+,- \qquad (2.8)$$

and may be interpreted as creation operators of P_0-scalar and P-scalar biphoton kinematic clusters respectively. For example, in the cases $m=1,2$ we have the following expressions [1,3]

$$a)|p\mu>=[(p-\mu)!(p+\mu)!]^{-1/2}(a^+_+(1))^{|\mu|+\mu}(a^+_-(1))^{|\mu|-\mu}(Y^+_{11})^{p-|\mu|}|0>,\quad n=2p, \qquad (2.9a)$$

$$b)|p,\mu;n,\lambda=t>=$$
$$[\frac{(2p+1)(p+\mu)!(p-\mu)!(p-t)!(p+t)!}{(n/2+p+1)!(n/2-p)!}]^{1/2}$$
$$\sum_\alpha\frac{(a^+_+(1))^{p+\mu-\alpha}(a^+_-(1))^{t-\mu+\alpha}(a^+_+(2))^\alpha(a^+_-(2))^{p-t-\alpha}}{(\alpha)!(p-t-\alpha)!(p+\mu-\alpha)!(t+\alpha-\mu)!}(X^+_{12})^{n/2-p}|0> \qquad (2.9b)$$

where $2t=n(1)-n(2)=n_+(1)+n_-(1)-n_+(2)-n_-(2)$ is the difference of the photon numbers in the first and second ST modes. In general, the states $|p,\mu;n,\lambda>$, whose explicit forms (in terms

of the $SU(2)$ generating invariants) can be found in [10,15,25], describe light beams representing a mixture of both usual photons and P- and P_0-scalar biphotons[1,2].

Biphoton operators X_{ij}, X_{ij}^+ generate the Lie algebra $so^*(2m)$ commuting with the polarization invariance algebra $su(2)_p = Span\{P_\alpha\}$[2]. Therefore, states $|\psi>$, belonging to a subspace $L(p\mu)$ of states with given p, μ at initial time, will evolve in this subspace $L(p\mu)$ under action of the interaction Hamiltonians

$$H_X = H^{'}_{int}(\{X_{ij}, X_{ij}^+; E_{ij}\}) \tag{2.10a}$$

describing some anisotropic parametric processes. Extending the algebra $so^*(2m)$ by adding operators Y_{ij}, Y_{ij}^+ we get the algebra $u(m,m)$ commuting with the polariztion subalgebra $u(1) = Span\{P_0\}$ and associated with interaction Hamiltonians

$$H_{X,Y} = H^{"}_{int}(\{Y_{ij}, Y_{ij}^+; X_{ij}, X_{ij}^+; E_{ij}\}) \tag{2.10b}$$

which keep invariant for time evolution subspaces $L^{'}(\mu) = \sum_{p \geq \|\mu|} L(p\mu)$ [10]. The algebra $u(m,m)$ contains the subalgebra $sp(2m, R)$ generated by biphoton operators Y_{ij}, Y_{ij}^+.

3 Generalized coherent states and quasiclassical description of light polarization

In this section, developing results [1,3,19], we examine in the $2m$-mode Fock space $L_F(2m)$ different types of polarization GCS associated with the $SU(2)_p$ group orbits and useful in applications.

As is well known, general GCS of the $SU(2)$ group orbit type are defined in accordance with (1.1) as follows [11]

$$|\xi; \psi_0\rangle \equiv |\theta, \varphi; \psi_0\rangle = D(g(\xi(\vec{n})))|\psi_0\rangle, D(g(\xi)) \equiv \exp(\xi J_+ - \xi^* J_-) \tag{3.1}$$

where $\xi(\vec{n}) = -\theta/2 \exp(-i\varphi)$, $\quad 0 \leq \theta \leq \pi, \quad 0 \leq \varphi \leq 2\pi$ are the angular coordinates of the unit vector $\vec{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ determining a position of the"classical" quasispin $\vec{J} = (J_\alpha)$ on its Poincaré sphere $S^2_P(\theta, \varphi)$; $|\psi_0\rangle$ is a certain reference vector in the space L of the states of the system. From the physical viewpoint, the states $|\xi; \psi_0\rangle$ describe output light beams obtained by means of action of quantum "$SU(2)$-rotators" with Hamiltonians

$$H_{SU(2)} = gJ_+ + g^* J_- \tag{3.2}$$

on the input beams in the quantum state $|\psi_0\rangle$ (see, e.g., [18,26,27] and references therein for possible realizations of such rotators in experimental devices).

For spin systems having a fixed spin value j one of the basis vectors $|jm\rangle$ of the irreducible representation (irrep) $D^j(SU(2))$ is used as $|\psi_0\rangle$, and the values of $m = \pm j$ correspond to the GCS most near to classical states [11]. A peculiarity of the polarization quasispin ($J_\alpha = P_\alpha$) of the light fields is that according to Eq. (2.5) the Fock spaces $L_F(2m)$ may be viewed as direct sums of the specific $SU(2)$ fiber bundles and contain the subspaces $L^{j\sigma}$ of the irrep $D^j(SU(2))$ with $j = p = 0, 1/2, 1, \ldots$, generally (for $m \geq 2$) with a certain multiplicity σ. Hence, in order to get the "complete polarization portrait" of the quantum light field and to calculate different physical quantities defined on the whole space $L_F(2m)$ one should have complete sets of the $SU(2)_p$ GCS

(3.1) with a set of reference vectors $|\psi_0\rangle = |\psi_0^{p,\sigma}\rangle \in L^{p,\sigma}, \quad p = 0, 1/2, \ldots$, or with a set of vectors $|\psi_0\rangle = |\psi_{0,\gamma}\rangle$, having nonzero projections on each of the subspaces $L^{P,\sigma}, \quad p = 0, 1/2, \ldots$ that, e.g., occurs for states describing physical light beams. Below we consider some examples of both types GCS in both mathematical and physical aspects.

In the first case, using the decomposition (2.5), it seems natural to choose for $|\psi_0^{p,\sigma}\rangle$ the vectors $|p\mu; n, \lambda\rangle$ from Eq. (2.6). Then, making use of the definition (3.1) and of the transformation properties of the operators $a^+_\pm(j), Y^+_{ij}, X^+_{ij}$ with respect to the $SU(2)_p$ group transformations $D(g(\xi))$ from (3.1) [1,10],

$$a) a^+_\pm(j) \longrightarrow \tilde{a}^+_\pm(j) \equiv (\eta^\pm(\theta,\varphi) a^+(j)) = a^+_\pm(j) \cos\frac{\theta}{2} \pm a^+_\mp(j) \exp(\pm i\varphi) \sin\frac{\theta}{2}, \tag{3.3a}$$

$$b) Y^+_{ij} \longrightarrow \tilde{Y}^+_{ij} = Y^+_{ij} \cos\theta + 1/2 \sin\theta [a^+_+(i) a^+_+(j) \exp(i\varphi) + a^+_-(i) a^+_-(j) \exp(-i\varphi)], \tag{3.3b}$$

$$c) X^+_{ij} \longrightarrow X^+_{ij} \tag{3.3c}$$

we get the sets $\{|\theta, \varphi; p, \mu, n, \lambda\rangle\}$ of the polarization GCS generated by the reference vectors $|p, \mu; n, \lambda\rangle$ and reproducing their form in terms of "$SU(2)$-rotated" operators (3.3) [1,3]; herewith the operator $(\eta^\pm(\theta,\varphi), a^+(j))$ may be interpreted as the creation operator of the elliptically polarized photon in the j-th ST mode having the ellipticity parameters determined by the angles θ, φ [28]. Evidently, using the definition (3.1) and Eq. (3.3) one can also obtain expansions of GCS $\{|\theta, \varphi; p, \mu, n, \lambda\rangle\}$ in terms of initial states $|p, \mu; n, \lambda\rangle$:

$$|\theta, \varphi; p, \mu, n, \lambda\rangle = \sum_{\mu'} u^p_{\mu'\mu}(\theta, \varphi) |p\mu'; n, \lambda\rangle,$$

$$u^p_{\mu'\mu}(\theta,\varphi) = [\frac{(p+\mu)!(p-\mu')!}{(p+\mu')!(p-\mu)!}]^{1/2} \frac{\exp(i\varphi(\mu-\mu'))}{(\mu-\mu')!} (\tan\frac{\theta}{2})^{\mu-\mu'} (\cos\frac{\theta}{2})^{2p}$$
$$F(\mu - p, -p - \mu'; \mu - \mu' + 1; -\tan^2\frac{\theta}{2}) \tag{3.4}$$

where the expansion coefficients $u^p_{\mu'\mu}(\theta,\varphi)$ are particular $SU(2)$ D-functions and $F(a, b; c; z)$ is the Gauss hypergeometric function.

The states $\{|\theta, \varphi; p, \mu, n, \lambda\rangle\}$ belong , from the mathematical point of view, to the class of *semi-coherent* ones (which are coherent (quasiclassical) in polarization degrees of freedom and orthonormalized (strongly quantum) in other ones) [1] since their overlap integral, easily calculated with the help of Eq. (3.4), has the form

$$\langle\theta, \varphi; p, \mu, n, \lambda|\theta', \varphi'; p', \mu', n', \lambda'\rangle \equiv I_{p,p';n,n';\mu,\mu';\lambda,\lambda'}(\eta^\pm, \eta'^\pm) =$$

$$\delta_{pp'}\delta_{nn'}\delta_{\lambda\lambda'} \sum_{\mu"} [u^p_{\mu"\mu}(\theta,\varphi)]^* u^p_{\mu"\mu'}(\theta',\varphi') = \delta_{pp'}\delta_{nn'}\delta_{\lambda\lambda'} [\frac{(p-\mu)!(p+\mu')!}{(p+\mu)!(p-\mu')!}]^{1/2} \times$$

$$\frac{(\eta'^+\eta^{+*})^{p+\mu} (\eta'^-\eta^{-*})^{p-\mu'} (\eta'^+\eta^{-*})^{\mu'-\mu}}{(\mu'-\mu)!} F(-p-\mu', -p+\mu; 1+\mu'-\mu; -|\frac{(\eta'^+\eta^{-*})}{(\eta'^+\eta^{+*})}|^2) \tag{3.5}$$

where $F(a, b; c; z)$ is the Gauss hypergeometric function and

$$(\eta'^+\eta^{+*}) = \cos\frac{\theta}{2}\cos\frac{\theta'}{2} + \sin\frac{\theta}{2}\sin\frac{\theta'}{2}\exp(i(\varphi'-\varphi) = (\eta'^-\eta^{-*})^*,$$

$$(\eta'^{-}\eta^{+*}) = -\exp(-i\varphi)'\cos\frac{\theta}{2}\sin\frac{\theta'}{2} + \sin\frac{\theta}{2}\cos\frac{\theta'}{2}\exp(-\varphi) = -(\eta'^{+}\eta^{-*})^{*} \qquad (3.5')$$

The GCS sets $\{|\theta, \varphi; p, \mu, n, \lambda\rangle\}$ contain subsets of states

$$|\theta, \varphi; p, n, \lambda\rangle_{\pm} \equiv \exp(\xi P_{+} - \xi^{*} P_{-})|p, \pm p; n, \lambda\rangle =$$

$$\sum_{\mu}[\frac{(2p)!}{(p+\mu)!(p-\mu)!}]^{1/2}(\pm\sin\frac{\theta}{2})^{p\mp\mu}(\cos\frac{\theta}{2})^{p\pm\mu}\exp[-i(\mu\mp p)\varphi]|p, \mu; n, \lambda\rangle \qquad (3.6)$$

which satisfy the maximal classicality criterion [11] in the polarization degrees of freedom and are (over)complete in $L_F(2m)$ yielding the following decomposition of the identity operator $\hat{I}$ [3,15]:

$$\hat{I} = \sum_{n,p,\lambda}\int_0^{\pi}\int_0^{2\pi}\frac{(2p+1)}{4\pi}\sin\theta d\theta d\varphi|\theta, \varphi; p, n, \lambda\rangle_{\pm}\langle\theta, \varphi; p, n, \lambda|_{\pm} \qquad (3.7)$$

that provides possibilites of calculating different physical averages on $L_F(2m)$ using these GCS. We also note that GCS (3.6) in their form are specific generating functions for states $|p, \mu; n, \lambda\rangle$ [15,25]; therefore, using states (3.6) as reference vectors in Eq. (3.1) (with another parameter $\xi(\vec{n'}) = -\theta'/2\exp(-i\varphi'))$ we obtain (after the substitution $\theta = \pi/2, \exp(-i\varphi) = z$) generating functions for GCS (3.4) that can be used in concrete calculations. Moreover, both sets $\{|\theta, \varphi; p, n, \lambda\rangle_{\pm}\}$ are equivalent from the mathematical viewpoint as it is seen from the formal equality

$$|\theta, \varphi; p, n, \lambda\rangle_{-} = \exp(-i2p\varphi)|\theta + \pi, \varphi; p, n, \lambda\rangle_{+} \qquad (3.6')$$

which follows from Eq. (3.6). Therefore, hereafter we will use, as a rule, only the set $\{|\theta + \pi, \varphi; p, n, \lambda\rangle_{+}\}$ omitting for the sake of simplicity the subscript " + ".

The construction of Eq. (3.6) is simplified in the picture of independent ST modes, when the group $SU(2)_p$ acts in the space $L_F(2)$ of each j-th ST mode independently, and its action is determined by the angles (θ_j, φ_j) characterzing by "partial" P-quasispin components $P_\alpha(j)$ and Poincaré spheres $S^2_P(j)$:

$$|\{\theta_j, \varphi_j\}; \{n_j\}\rangle_{\pm} \equiv \prod_{j=1}^{m}\exp(\xi_j P_{+}(j) - \xi_j^{*} P_{-}(j))(a^{+}_{\pm}(j))^{n_j}[n_j!]^{-1/2}|0\rangle = \prod_{j=1}^{m}\frac{(\eta^{\pm}(\theta_j, \varphi_j), a^{+}(j))^{n_j}}{[n_j!]^{1/2}}|0\rangle \qquad (3.8)$$

The set of GCS (3.8) is complete (an analog of Eq. (3.7) is valid for it) and yields the "polarization phase portrait of the field" adequate to independent measurements for each ST mode. The connection between the sets (3.6) and (3.8) is realized via the generalized Clebsh-Gordan coefficients of $SU(2)_p$ [15]. Note that choosing arbitrary Fock states $\prod_{j=1}^{m}(a^{+}_{+}(j))^{n_j^{+}}(a^{+}_{-}(j))^{n_j^{-}}|0\rangle$ as the reference vector in (3.1) we obtain polarization GCS

$$|\{\theta, \varphi\}; \{n_j^{+}; n_j^{-}\}\rangle = \prod_{j=1}^{m}\frac{(\eta^{-}(\theta, \varphi), a^{+}(j))^{n_j^{-}}(\eta^{+}(\theta, \varphi), a^{+}(j))^{n_j^{+}}}{[n_j^{+}!n_j^{-}!]^{1/2}}|0\rangle \qquad (3.8^{*})$$

which are similar to states (3.8) in their form and may be used for expanding other kinds of polarization GCS in series using appropriate expansions of the reference vectors in terms of the Fock states. The GCS (3.8) are the Fock states in terms of the "rotated" photon operators

$(\eta^{\pm}(\theta,\varphi), a^+(j))$ and are unitarily equivalent to to the initial Fock states; hence there are some difficulties to produce them (as well as states $|\theta,\varphi;p,\mu,n,\lambda\rangle)$ in physical experiments[13]. Therefore, from the physical viewpoint it is of interest to consider other types of GCS of $SU(2)_p$, which do not contain the discrete parameters n,λ labeling $SU(2)$-invariant subspaces $L^{(p,\sigma=n,\lambda}$. So, if taking in (3.1) the reference vectors $|\psi_0^{p,\sigma}\rangle$ of the form[1]

$$|\psi_0^{p,\{\zeta_{ij},\kappa_{ij}\}}\rangle_{\pm} =$$

$$\exp(\sum_{i<j}[\kappa_{ij}E_{ij} + \zeta_{ij}X_{ij}^{+} - \kappa_{ij}^{*}E_{ji} - \zeta_{ij}^{*}X_{ij}])(a_{\pm}^{+}(1))^{2p}[(2p)!]^{-1/2}|0\rangle, E_{ij} \equiv \sum_{\alpha=\pm} a_{\alpha}^{+}(i)a_{\alpha}(j) \qquad (3.9)$$

we obtain states

$$|p;\{\zeta_{ij},\kappa_{ij}\};\theta,\varphi\rangle_{\pm} =$$

$$\exp(\sum_{i<j}[\kappa_{ij}E_{ij} + \zeta_{ij}X_{ij}^{+} - \kappa_{ij}^{*}E_{ji} - \zeta_{ij}^{*}X_{ij}])(\eta^{\pm}(\theta,\varphi), a_{\pm}^{+}(1))^{2p}[(2p)!]^{-1/2}|0\rangle \qquad (3.10)$$

which are GCS with respect to both the polarization $SU(2)_p$ and the "biphoton" $SO^*(2m)$ groups, (over)complete in $L_F(2m)$, provide a quasiclassical description of both polarization and biphoton degrees of freedom[10] and are generated in processes governed by Hamiltonians (2.10a) and (3.2). Note that using the "disentangling theorems"[11,29] for the $SO^*(2m)$ displacement operators $\exp(\sum_{i<j}[\kappa_{ij}E_{ij} + \zeta_{ij}X_{ij}^{+} - \kappa_{ij}^{*}E_{ji} - \zeta_{ij}^{*}X_{ij}])$ one can obtain expansions of states (3.10) in series of states (3.6). For example, in the simplest non-trivial case $m=2$, when $SO^*(4) = SU(2)\otimes SU(1,1)$, $SU(2) = Span\{J_+ = E_{12}, J_- = E_{21}, J_0 = 1/2(E_{11}-E_{22})\}, SU(1,1) = Span\{K_+ = X_{12}^{+}, K_- = X_{12}, K_0 + = 1/2(E_{11}+E_{22})+1\}$, with the help of results [11,29,15] we find [1]

$$|p;\{\zeta,\kappa\};\theta,\varphi\rangle_{\pm} = \exp([\kappa E_{12} - \kappa^{*}E_{21}])\exp([\zeta X_{12}^{+} - \zeta^{*}X_{12}])(\eta^{\pm}(\theta,\varphi), a_{\pm}^{+}(1))^{2p}[(2p)!]^{-1/2}|0\rangle =$$

$$[\cosh|\zeta|]^{-2(p+1)}[\cos|\kappa|]^{2p}\sum_{T,\tau}(\tanh|\zeta|\exp(i\arg\zeta))^{T}(-\tan|\kappa|\exp(-i\arg\kappa))^{\tau}$$

$$[\frac{(T+2p+1)!}{(2p+1)(T)!(2p-\tau)!\tau!}]^{1/2}|\theta,\varphi;p,n=2(T+p),t=p-\tau\rangle_{\pm} \qquad (3.11)$$

From the physical point of view it is also of interest to consider generalizations of GCS (3.10) related to Hamiltonians (2.10b) and (3.1) and obtained when replacing the reference vector (3.9) by the $U(m,m)$ group GCS

$$|\psi_0^{p,\{\zeta_{ij},\kappa_{ij},\gamma_{ij}\}}\rangle_{\pm} =$$

$$\exp(\sum_{i<j}[\kappa_{ij}E_{ij} + \zeta_{ij}X_{ij}^{+} + \gamma_{ij}Y_{ij}^{+} - \kappa_{ij}^{*}E_{ji} - \zeta_{ij}^{*}X_{ij} - \gamma_{ij}^{*}Y_{ij}])(a_{\pm}^{+}(1))^{2p}[(2p)!]^{-1/2}|0\rangle, \qquad (3.12)$$

Without dwelling on a detailed analysis of such GCS we write down their expressions (cf. (2.9a))

$$|p;\gamma;\theta,\varphi\rangle_{\pm} = \exp(\gamma\tilde{Y}_{11}^{+} - \gamma^{*}\tilde{Y}_{11})\frac{(\tilde{a}_{\pm}^{+})^{2p}}{\sqrt{(2p)!}}|0\rangle =$$

$$[\cosh|\gamma|]^{-(2p+1)}\sum_{\tau}(\tanh|\gamma|\exp(i\arg\gamma))^{\tau}[\frac{(2p+\tau)!}{(2p)!\tau!}]^{1/2}|\theta,\varphi;p,\mu=\pm p,n=2(\tau+p)\rangle \qquad (3.13)$$

for the states which are GCS of $SU(2)_p \otimes SU(1,1)$, $SU(1,1) = Span\{K'_+ = Y^+_{11}, K'_- = Y_{11}, K'_0 + = 1/2(E_{11}+1)\}$ and describe in the case of $p = 0$ the twin-photon beams of unpolarized light obtained in degenerate parametric processes [2,10,23].

An alternate type of "physical" polarization GCS may be obtained if one takes in (3.1) the sets of reference vectors $|\psi_{0,\gamma}\rangle$, having the nonzero projections on all $L^{p,\sigma}$. A natural example of such a set is the familiar set of Glauber's coherent states:

$$|\{\alpha^+_j, \alpha^-_j\}\rangle = \prod_{j=1}^{m} \exp[\alpha^+_j a^+_+(j) + \alpha^-_j a^+_-(j) - (\alpha^+_j)^* a_+(j) - (\alpha^-_j)^* a_-(j)]|0\rangle. \tag{3.14}$$

Then using the definition (3.1) and the $SU(2)_p$ transformation properties (3.3) of $a^\pm_\pm(j)$, one gets from Eq. (3.14) the set of GCS

$$|\theta, \varphi; \{\alpha^+_j, \alpha^-_j\}\rangle \equiv \exp(\xi P_+ - \xi^* P_-)|\{\alpha^+_j, \alpha^-_j\}\rangle = |\{\tilde{\alpha}^+_j(\theta, \varphi), \tilde{\alpha}^-_j(\theta, \varphi)\}\rangle,$$

$$\tilde{\alpha}^\pm_j(\theta, \varphi) = \alpha^\pm_j \cos\frac{\theta}{2} \mp \exp(\mp i\varphi)\alpha^\mp_j \sin\frac{\theta}{2}, \tag{3.15}$$

which can be obtained expirementally by action of quantum polarization "rotators" on the initial states (3.14).

The states (3.15) are analogous to the initial set (3.14), but with two extra (redundant from the mathematical viewpoint) parameters involved, namely, θ and φ. This redundance can be removed by imposing two constraints on parameters $\alpha^\pm_j$ in (3.14). For example, in the case of $m = 1$ one may choose the subsets of (3.15) in the form (j is fixed):

$$|\theta_j, \varphi_j; \alpha^+_j\rangle_+ \equiv |\theta_j, \varphi_j; \alpha^+_j, 0\rangle = |\alpha^+_j \cos\frac{\theta_j}{2}, \alpha^+_j \exp(i\varphi_j) \sin\frac{\theta_j}{2}\rangle, \quad |\theta_j, \varphi_j; \alpha^-_j\rangle_- \equiv |\theta_j, \varphi_j; 0, \alpha^-_j\rangle, \tag{3.16}$$

which describe the elliptically polarized waves and coincide with usual Glauber CS (3.14) for $m = 1$ but with picking out polarization coordinates θ_j, φ_j explicitly (unlike the form (3.14)) that it is important, e.g., for constructing polarization Q-functions [3,19]. In the general case $m \geq 2$ the subsets (3.16) are not complete in $L_F(2m)$ that, however, is unimportant from the physical point of view. (The complete sets of GCS of such a type in $L_F(2m)$ may be obtained, e.g., by taking m-fold product of GCS (3.16) [3,19].) Another possibility is related to subsets of GCS (3.15) where states (3.14) are constrained by conditions

$$\sum_{j=1}^{m} arg\alpha^+_j = 0, \quad \sum_{j=1}^{m} arg\alpha^-_j = 0 \tag{3.17}$$

which determine for $m = 1$ an alternate to (3.16) set of polarization GCS. Note that from the physical point of view one can also determine other types of "physical" polarization GCS obtained via actions of "polarization rotators" on different physical input states , e.g., eigenstates of the biphoton destruction operators Y_{ij}, X_{ij}, etc. [1,30] that, however, is beyond the scope of the paper.

The sets of GCS obtained above may be used for the quasiclassical analysis of the polarization properties of quantum light fields. In particular, one can use the definition (1.2) to introduce the complete polarization Q-functions [3] as follows

$$Q(\theta, \varphi; \psi_0; \rho) \equiv \mathrm{Tr}[\rho|\theta, \varphi; \psi_0\rangle\langle\theta, \varphi; \psi_0|] = \langle\theta, \varphi; \psi_0|\rho|\theta, \varphi; \psi_0\rangle, \tag{3.18}$$

where ρ is the complete density operator for the state of the field, $|\theta, \varphi; \psi_0\rangle$ being defined by Eq. (3.1). Then, substituting the specifications (3.6), (3.8), (3.10), (3.12), (3.15) and (3.16) for $|\theta, \varphi; \psi_0\rangle$ into Eq. (3.18), we get the appropriate concrete types of the complete polarization quasiprobability functions. Note, however, that such functions, besides the dependence on the polarization parameters θ, φ, involve the additional quantum numbers $n, \lambda, \{\alpha_j^{\pm}\}$, etc., which characterize the non-polarization degrees of freedom of the field. Therefore, to get its "pure polarization quasiclassical portrait" in the ρ-state it is sufficiently to make use of the reduced polarization quasiprobability functions $Q^p(\varphi, \theta; \psi_0; \rho)$, resulting from Eq. (3.18) after the summation (or integration) over non-polarization variables. Such functions determine "error bodies" and may be used to analyse the "polarization squeezing" [10] in analogy with the familiar Q-functions in the case of the standard quadrature squeezing [6]. Keeping in mind the completeness of GCS (3.6), one may determine for this aim only one type of Q^p-functions based on GCS (3.6):

$$Q^p(\theta, \varphi; p; \rho) = \sum_{n,\lambda} \langle \theta, \varphi; p, n, \lambda | \rho | \theta, \varphi; p, n, \lambda \rangle, \tag{3.18*}$$

Let us calculate some of such Q^p-functions, substituting in (3.18*) concrete density operators $\rho_i = |\theta', \varphi'; \psi_0^i\rangle\langle\theta', \varphi'; \psi_0^i|$ for pure states $|\theta', \varphi'; \psi_0^i\rangle, i = 1, 2$, described by Eqs. (3.4) and (3.15) (and restricting oneself for the sake of simplicity by the case of $m = 2$ in (3.15)). Then, after some algebra one gets the following expressions for the appropriate Q^p-functions

$$a) Q^p(\theta, \varphi; p; \rho_1) = \delta_{pp'} \, [\frac{(2p)!}{(p+\mu')!(p-\mu')!}] |(\eta'^{+}\eta^{+*})|^{2(p+\mu')} |(\eta'^{-}\eta^{+*})|^{2(p-\mu')} =$$

$$[\cos^2 \frac{\theta - \theta'}{2} - \sin^2 \frac{\varphi' - \varphi}{2} \sin\theta \sin\theta']^{p+\mu'} [\sin^2 \frac{\theta - \theta'}{2} + \sin^2 \frac{\varphi' - \varphi}{2} \sin\theta \sin\theta']^{p-\mu'}, \tag{3.19a}$$

$$b) Q^p(\theta, \varphi; p; \rho_2) = \frac{\exp(-\sum_{i=1,2}[|\alpha_i^+|^2 + |\alpha_i^-|^2]) \, (2p+1) J_{1+2p}(-2|[\alpha_1\alpha_2]|)}{|[\alpha_1\alpha_2]|^{2p+1}} \times$$

$$[(\sum_{i=1,2} |\alpha_i^+|^2) |(\eta'^{+}\eta^{+*})|^2 + (\sum_{i=1,2} |\alpha_i^-|^2) |(\eta'^{-}\eta^{+*})|^2 + 2Re[(\sum_{i=1,2} \alpha_i^+ \alpha_i^{-*})(\eta'^{+}\eta^{+*})(\eta'^{-}\eta^{+*})^*]]^{2p},$$

$$[\alpha_1\alpha_2] = \alpha_1^+\alpha_2^- - \alpha_1^-\alpha_2^+ \tag{3.19b}$$

where $J_{1+2p}(-2|[\alpha_1\alpha_2]|)$ is the Bessel function. Note that is due to the structure of Eqs (3.11), (3.13) angular dependences of $Q^p(\theta, \varphi; p; \rho_i), i = 3, 4$, for states described by these equations can be easily obtained from (3.19a). For comparison we also write down the Q^p-function

$$e) Q^p(\theta, \varphi; p; \rho_{th}(1)) = [1 - \exp(-\beta)]^2 \exp(-2p\beta),$$

$$\rho_{th}(1) = [1 - \exp(-\beta)]^2 \sum_{n,\mu} \exp(-n\beta) |p = n/2\mu >< p = n/2\mu|, \quad \beta = (kT)^{-1} \tag{3.20}$$

for the case of a single ST mode in the mixed state $\rho_{th}(1)$ [13] of the thermal equilibrium.

Similarly, one can determine Q^p- representations

$$f(\{P_\alpha\}; \theta, \varphi; \psi_0) \equiv \langle \theta, \varphi; \psi_0 | f(\{P_\alpha\}) | \theta, \varphi; \psi_0 \rangle \tag{3.21}$$

for arbitrary polarization operators $f(\{P_\alpha\})$ using for this aim polarization characteristic functions [11]

$$\chi^{\psi_0}_{\{P_\alpha\}}(\{\nu_i\}) = \langle\theta,\varphi;\psi_0|\prod_i \exp(\nu_i P_i)|\theta,\varphi;\psi_0\rangle \tag{3.22}$$

where exponents are taken in an order. Calculations of the (3.22) right sides, evidently, are reduced to finding overlap integrals like Eq. (3.5). Without dwelling on a detailed discussion of this question we find some characteristic functions which are useful in applications. In particular, in studies of squeezing problems it is necessary to have Q- representations for lowest powers of P_α (see,e.g., [5-10] and the following Section). Keeping also in mind expansions of the type (3.11), (3.13) it is sufficiently for this aim to calculate particular characteristic functions $\chi^{\psi_0}_{P_\alpha}(\mu_\alpha)$ (with α being fixed) for GCS (3.4) and (3.15) (or (3.14)). Then, using the definitions $2P_1 = (P_+ + P_-), 2P_2 = i(P_+ - P_-)$ and Eqs. (3.3)- (3.5), (3.15) one finds

$$a)\chi^{sc_{p,\mu}}_{P_{\alpha=1,2}}(\nu_{k=1,2}) \equiv \langle\theta,\varphi;p,\mu;n,\lambda|\exp(\nu_k P_k)|\theta,\varphi;p,\mu;n,\lambda\rangle =$$

$$\sum_\alpha \frac{(p+\mu)!(p-\mu)!}{(p+\mu-\alpha)!(p-\mu-\alpha)!\alpha!\alpha!}[\sin^2\frac{\tau}{2}(\sin^2\theta\sin^2(\frac{\pi k}{2}-\varphi)-1)]^\alpha \times$$

$$[\cos\frac{\tau}{2}+i\sin\frac{\tau}{2}\sin\theta\sin(\frac{\pi k}{2}-\varphi)]^{p+\mu-\alpha}[\cos\frac{\tau}{2}-i\sin\frac{\tau}{2}\sin\theta\sin(\frac{\pi k}{2}-\varphi)]^{p-\mu-\alpha}, \tag{3.23a}$$

$$b)\chi^{Gcs}_{P_{\alpha=1,2}}(\nu_{k=1,2}) \equiv \langle\{\alpha^+_j(\theta,\varphi),\alpha^-_j(\theta,\varphi)\}|\exp(\nu_k P_k)|\{\alpha^+_j(\theta,\varphi),\alpha^-_j(\theta,\varphi)\}\rangle =$$

$$\exp\left[(\cos\frac{\tau}{2}-1)\sum_{j=1}^m\{|\alpha^+_j(\theta,\varphi)|^2+|\alpha^-_j(\theta,\varphi)|^2\}\right]$$

$$\times\exp\left[i\sin\frac{\tau}{2}\sum_{j=1}^m 2Im\{\alpha^-_j(\theta,\varphi)(\alpha^+_j(\theta,\varphi))^*\exp(\frac{-i\pi k}{2})\}\right], \tag{3.23b}$$

where $\nu_k = i\tau$, τ is purely real and $k = 1(2)$ for $P_{1(2)}$. Similarly, using a diagonal analog of the transformations (3.3), one gets

$$a)\chi^{sc_{p,\mu}}_{P_0}(\nu_0 = i\tau) = \sum_\alpha \frac{(p+\mu)!(p-\mu)!}{(p+\mu-\alpha)!(p-\mu-\alpha)!\alpha!\alpha!}[-\sin^2\frac{\tau}{2}\sin^2\theta]^\alpha \times$$

$$[e^{\frac{-i\tau}{2}}\sin^2\frac{\theta}{2}+e^{\frac{i\tau}{2}}\cos^2\frac{\theta}{2}]^{p+\mu-\alpha}[e\frac{i\tau}{2}\sin^2\frac{\theta}{2}+e^{\frac{-i\tau}{2}}\cos^2\frac{\theta}{2}]^{p-\mu-\alpha}, \tag{3.24a}$$

$$b)\chi^{Gcs}_{P_0}(\nu_0 = i\tau) =$$

$$\exp\left[-\sum_{j=1}^m\{|\alpha^+_j(\theta,\varphi)|^2+|\alpha^-_j(\theta,\varphi)|^2\}\right]\exp\left[\sum_{j=1}^m\{|\alpha^+_j(\theta,\varphi)|^2\exp(i\tau)+|\alpha^-_j(\theta,\varphi)|^2\exp(-i\tau)\}\right], \tag{3.24b}$$

4 Squeezing and a new classification of polarization states of light in quantum optics

The decomposition (2.5) and the results obtained in the previous Section yield an effective tool for studies of the squeezing problems of multimode light beams with consideration of polarization that, in turn, implies a new classification of the polarization states of quantum light fields [10]. In fact, a definition of squeezing in quantum mechanics is based on an analysis of different uncertainty relations for a set $\{A_i, i = 1, ..., r > 1\}$ of non-commuting Hermitian operators A_i representing some quantum observables [5-10,31-34]. These relations are associated with specific measures of admissible quantum fluctuations ("noises") for observables A_i in the state $| >$ expressed in terms of expectations $< |(A_i)^s| >$ characterizing differences between quantum observables A_i and their classical analogs[11,12,31-33]. For example, the most widespread uncertainty relation (of the Weyl-Heisenberg type) has the form [11]

$$\Delta A_i \Delta A_j \geq 1/2| < |[A_i, A_j]| > | \tag{4.1}$$

where $(\Delta A)^2 \equiv \sigma_A =< |(A)^2| > -(< |A| >)^2$ is a standard quadratic measure (variance) of a deviation of the quantum quantity A from its classical analog ($< |A| >$). Specifically, for the single-mode electromagnetic field one makes use of two quadrature components of the field $A_1 = (a^+ + a)/\sqrt{2}, A_2 = i(a^+ - a)/\sqrt{2}$ as observables A_i, and $| < |[A_1, A_2]| > | = 1$ determines a boundary (vacuum or zero-point) level of admissible quantum field fluctuations [5]. Then conditions *a)* $S_{12}^A \equiv \Delta A_1 \Delta A_2 \longrightarrow$ *min with constraints (4.1) (a joint quasiclassical behaviour of* A_1 *and* A_2*)* and *b)* ΔA_1 *(or* ΔA_2*)* $< [(S_{12}^A)_{min}]^{1/2}$ *(a suppression of one quadrature noise)* define the usual one-mode field quadrature squeezing realized with the help of GCS $\exp(za^{+2} - z^* a^2)|\psi >$ of the group $SU(1,1) \sim Sp(2, R) = Lin\{L_0 = a^+ a/2 + 1/4, L_+ = (a^+)^2/2, L_- = a^2/2\}$ conserving the canonical commutation relation $[a, a^+] = 1$ and (when z is real) the "error area" S_{12}^A [5-7]. (Emphasize an importance of the condition a) in the definition above and of the S_{12}^A invariance with respect to $\exp(z[a^{+2} - a^2])$ (z is real) for a search of squeezed states; a relaxation of the first requirement leads to a "soft" (with a fixed value of S_{12}^A) squeezing notion whereas a rejection of both ones contradicts principal original ideas [5,6] and makes the class of squeezed states too large.)

However, for multimode fields the situation becomes more complicated as in this case we have a more vast set of observables which obey non-trivial commutation relations, and there exist many possibilities of definition of squeezing related to different choices (from physical considerations) of some subsets of observables, adequate joint uncertainty measures for them and some boundary (or reference) levels of admissible quantum fluctuations [8-10,34]. Specifically, in polarization quantum optics as such subsets, besides different field quadrature components [8,9], one may also take components of the P-quasispin obeying the commutation relations of the $su(2)_p$ algebra and subsets of unpolarized biphoton operators of X- and Y- types generating the $so^*(2m) \subset u(m, m) \supset sp(2m, R)$ algebras associated with Hamiltonians (2.10). That enables to define (when maintaining basic features of the concept above) different sorts of multimode light squeezing related to appropriate degrees of freedom. Without dwelling on all aspects of this vast topic we focus here our attention on features (including definitions) of specific kinds of squeezing related to polarization (and partially to biphoton) degrees of freedom applying for this purpose standard

uncertainty measures and their general analysis for arbitrary Lie algebras [11-12] as well as the GCS techniques of the algebras above.

First of all we note that the conditions (4.1) are less restrictive for generators A_i of arbitrary Lie algebras g than for one-mode field quadrature components since, in general, right sides of these inequalities are not fixed c-numbers but depend on quantum states under consideration. Furthermore, adequate measures of a joint quasiclassical behaviour of the set $\{A_i\}$ are the g-invariant quantities $(\Delta A)^2 = \sum g_{ij}[< A_iA_j > - < A_i >< A_j >]$ (g_{ij} is the Killing-Cartan metric tensor) related to Casimir operators of g rather than S^A_{ij} [11,12]. Therefore, in this case the squeezing definition above has to be modified (when retaining its basic features), e.g., as *a suppression of one or more "partial noises" ΔA_i at a minimal or a given reference level of the "collective noise" $(\Delta A)^2$ consistent with conditions (4.1).* A natural search of appropriate squeezed states may be realized in a set of the g GCS conserving values $(\Delta A)^2$ and the structure relations of g that provides (together with the modified definition) a group-theoretical treatment of the squeezing concept for $\{A_i\} \subset g$.

For example, a "purely polarization" squeezing is defined in such a manner by means of a minimization of a $SU(2)$-invariant "radial" uncertainty measure $(\Delta P)^2 \equiv \sum_\alpha \sigma^P_\alpha = \bar{p}(\bar{p}+1) - [degP < |N| > /2]^2$ of total polarization noises or its normalized version $(\delta P)^2 = (\Delta P)^2/(< |N| >)^2$ (determinig a level of polarization quasiclassicality of the field in a given pure quantum state $| >$) together with an analysis of the relations (4.1) for $A_i = P_i, i = 1,2,0$. For fixed values p of the polarization quasispin it may be realized on GCS (3.6) (or on their linear combinations over discrete parameters n, λ) [10]. Furthermore, owing to the complementarity (commutativity of actions) of the algebras $so^*(2m)$ and $su(2)_p$ on $L_F(2m)$ (cf. (2.8)) this definition is completely compatible with the "X-biphoton" squeezing defined similarly (but with peculiarities due to a $(\Delta X)^2$ definition) for observables of the $so^*(2m)$ algebra, and, according to the analysis [11,12], GCS (3.10) realize such a joint "polarization - X-biphoton" squeezing; therefore, operators $S_P(\{\xi\}) \equiv \exp(\xi P_+ - \xi^* P_-)$ and $S_X(\zeta_{ij}\}) \equiv \exp(\sum_{i<j}[\zeta_{ij}X^+_{ij} - \zeta^*_{ij}X_{ij}])$ may be called, respectively, as polarization and X-biphoton squeezed operators. An extra peculiarity follows in polarization squeezing studies from the fact that in accordance with Eq. (2.5) physical states describing light beams do not belong to a single irreducible subspace of $su(2)_p$; besides, in polarization optics there exist only two different basic measurement procedures related to linear (P_1 or P_2) and circular (P_0) polarization types. Bearing in mind these general remarks we examine below different polarization GCS of the previous Section as test functions (with p, θ, φ being variables) to determine different (related to possible specifications of a suppression of "partial noises" ΔP_i) types of polarization squeezed states.

By analogy with the usual one-mode quadrature squeezing we consider at first polarization noises for the Glauber's GCS (3.15). Using the characteristic functions (3.23b) and (3.24b) one finds relations

$$a)(\Delta P_1)^2 = (\Delta P_2)^2 = (\Delta P_0)^2 = 1/4\sum_{j=1}^{m}\{|\alpha^+_j|^2 + |\alpha^-_j|^2\} =< |N| > /4,$$

$$(\Delta P)^2 = 3 < |N| > /4, \quad (\delta P)^2 = \frac{3}{4 < |N| >} \tag{4.2a}$$

$$b)| < |P_0| > | = 1/2|\sum_{j=1}^{m}\{[|\alpha^+_j|^2 - |\alpha^-_j|^2]\cos\theta - 2Re[\alpha^-_j\alpha^{+*}_j\exp(-i\varphi)]\sin\theta\}|,$$

$$| < |P_2| > | =$$

$$1/2|\sum_{j=1}^{m}\{[|\alpha_j^+|^2 - |\alpha_j^-|^2]\sin\theta\sin\varphi + 2Im[\alpha_j^-\alpha_j^{+*}]\cos^2\frac{\theta}{2} + 2Im[\alpha_j^-\alpha_j^{+*}e^{-i2\varphi}]\sin^2\frac{\theta}{2}\}|,$$

$$| < |P_1| > | =$$

$$1/2|\sum_{j=1}^{m}\{[|\alpha_j^+|^2 - |\alpha_j^-|^2]\sin\theta\cos\varphi + 2Re[\alpha_j^-\alpha_j^{+*}]\cos^2\frac{\theta}{2} + 2Re[\alpha_j^-\alpha_j^{+*}e^{-i2\varphi}]\sin^2\frac{\theta}{2}\}| \qquad (4.2b)$$

manifesting an absence of any polarization squeezing because all partial polarization noises $(\Delta P_\alpha)^2$ are equal to a quarter of the standard (Poissonian) quantum level $< |N| >$ of noises in optics [5-7] for any values of both polarization (φ, θ) and coherent $(\alpha_j^\pm)$ parameters; this feature distinguishes actions of polarization squeezed operators $S_P(\{\xi\})$ on GCS (3.14) from those for usual quadrature squeezed operators $\exp(za^{+2} - z^*a^2)$ (cf. [5-7]).

However, the situation is quite different for GCS (3.6), (3.10). Indeed, according to Eqs. (3.23a) and (3.24a), these GCS with a fixed value p minimize the uncertainty measure $(\Delta P)^2$:

$$a)(\Delta P)^2 = (\Delta P)^2_{min} = p \ \leq < N > /2, \quad (\delta P)^2 \leq \frac{1}{2 < N >} \qquad (4.3a)$$

and relations (4.1) for $A_i = P_i, i = 1, 2, 0$ have on these GCS the form:

$$b)\Delta P_1\Delta P_2 = p/2\ [\cos^2\theta + \frac{1}{4}\sin^4\theta\sin^2 2\varphi]^{1/2} \geq 1/2| < |P_0| > | = p/2\ |\cos\theta|,$$

$$\Delta P_1\Delta P_0 = p/2\ |\sin\theta|[1 - (\sin\theta\cos\varphi)^2]^{1/2} \geq 1/2| < |P_2| > | = p/2\ |\sin\theta\sin\varphi|,$$

$$\Delta P_2\Delta P_0 = p/2\ |\sin\theta|[1 - (\sin\theta\sin\varphi)^2]^{1/2} \geq 1/2| < |P_2| > | = p/2\ |\sin\theta\cos\varphi| \qquad (4.3b)$$

As is seen from Eqs. (4.2b), for $\theta = 0, \pi$ and any φ we get a complete suppression of the circular polarization noise $\Delta P_0 : \Delta P_0 = 0$ whereas $(\Delta P_1)^2 = (\Delta P_2)^2 = p/2$; therefore, GCS (3.6), (3.10) for $\theta = 0, \pi$ manifest a circular polarization squeezing unlike the circular coherent states (3.15) characterized by the condition: $\sum_{j=1}^m \alpha_j^-\alpha_j^{+*} = 0$. Similarly, GCS (3.6), (3.10) for $\theta = \pi/2, \varphi = 0 or \pi/2$ manifest a linear polarization squeezing. Another ("circular-linear") type of polarization squeezing may be determined by imposing the constraints:

$$(\Delta P_a)^2 \leq \ (\Delta P)^2/3 = p/3 \leq < N > /6, \quad a = 0, 1 \qquad (4.4a)$$

that corresponds to a joint suppression of one linear (ΔP_1) and the circular (ΔP_0) partial noises at the expense of increasing ΔP_2. As it follows from Eqs. (4.3), (4.4a) such a situation is attained for parameters φ, θ constrained by the conditions

$$\frac{1}{3} \leq \ \sin^2\theta\cos^2\varphi \leq \ \frac{4}{9}, \qquad \frac{1}{2} \leq \ \sin^2\theta \leq \ \frac{2}{3}, \quad \frac{1}{2} \leq \ \cos^2\varphi \leq \ 1 \qquad (4.4b)$$

However, in practice, it is more or less easy to produce only GCS (3.10) with $p = 0$ which provide an absolute polarization squeezing for any θ, φ and describe an absolutely unpolarized light ("polarization vacuum") characterized by relations [2,10]

$$< |(P_\alpha)^s| >= 0, \quad \alpha = 0, 1, 2, \ s \geq 1 \qquad (4.5a)$$

showing the full absence of appropriate polarization noises ($<|(P_\alpha)^s|> - (<|(P_\alpha)|>)^s$) of any order measured by appropriate noises of difference photocurrents in schemes [23]. (For comparison we point out that for usual quantum spin systems characterized by a fixed value j of angular momentum such an absolute squeezing for the $su(2)$ algebra observables is realized on the only vector $|j=0, m=0>$.) Note that for the case $m=2$ Eqs. (4.5a) imply relations

$$<|(P_\alpha(1))^s|> = (-1)^s <|(P_\alpha(2))^s|>,$$

$$<|(P_\alpha(1))^s|><|(P_\beta(1))^s|> = <|(P_\alpha(2))^s|><|(P_\beta(2))^s|> \qquad (4.5b)$$

that it is of interest for studies of the EPR-paradox, entangled states and "hidden variable" theories [20-22]. Besides, the polarization vacuum is the only sort of quantum light having the property $(\Delta P)^2 = 0$ as it follows from Eqs. (2.4b) and (4.1). The situation is somewhat different for GCS (3.12) also realized with the help of parametric oscillator generators (but corresponding to the Hamiltonians (2.10b)). In fact, as the $u(m,m) \supset sp(2m,R)$ algebras commute only with the $u(1) = Span\{P_0\}$ subalgebra of $su(2)_p$, the polarization ($S_P(\{\xi\})$) and Y-biphoton ($S_Y(\{\gamma_{ij}\}) \equiv \exp[\sum(\gamma_{ij}Y_{ij}^+ - \gamma_{ij}^* Y_{ij})]$) squeezed operators act in general not independently (as it follows from Eq. (3.3)). Therefore, for the particular case of GCS (3.13) generated by biphotons of the Y -type Eqs. (4.3) are replaced by relations

$$a)(\Delta P)^2 = p + \frac{1}{2}(p+1)(2p+1)\sinh^2|2\gamma| = \frac{(1-2p)[<N>(p+1)+p] + <N>^2/2}{2p+1}, \qquad (4.6a)$$

$$b)\Delta P_1 \Delta P_2 = \frac{1}{2}(\Delta P)^2[\cos^2\theta + \frac{1}{4}\sin^4\theta\sin^2 2\varphi]^{1/2} \geq 1/2|<|P_0|>| = 1/2p|\cos\theta|,$$

$$\Delta P_1 \Delta P_0 = \frac{1}{2}(\Delta P)^2[1-\sin^2\theta\cos^2\varphi]^{1/2}|\sin\theta| \geq 1/2|<|P_2|>| = 1/2p|\sin\theta\sin\varphi|,$$

$$\Delta P_2 \Delta P_0 = \frac{1}{2}(\Delta P)^2[1-\sin^2\theta\sin^2\varphi]^{1/2}|\sin\theta| \geq 1/2|<|P_1|>| = 1/2p|\sin\theta\cos\varphi| \qquad (4.6b)$$

depending on both polarization (φ, θ) and biphoton (γ) parameters. Evidently, repeating main steps of the Eqs. (4.3) analysis, one may determine "soft" (because of differences between Eq. (4.6a) and Eq. (4.3a)) versions of different types of polarization squeezing considered above. Specifically, for $\gamma \neq 0$ (determined, e.g., by conditions of squeezing biphoton or quadrature variables) Eq. (4.6a) is minimized when $p=0$ on states describing another (in comparison with GCS (3.10)) class of unpolarized quantum light (so-called twin-photon light with hidden polarization [26]) for which

$$(\Delta P)^2 = \frac{1}{2}\sinh^2|2\gamma| = <N> + <N>^2/2, \quad (\delta P)^2 = \frac{1}{<N>} + 1/2 \qquad (4.6a^*)$$

and Eqs. (4.5a) are valid only either for $s=1$, any α or for $\alpha = 0, \theta = 0, \pi$, any s, φ [2,10].

So, we have shown that polarization GCS generated by means of actions of squeezed operators $S_P(\{\xi\})$ $S_X(\zeta_{ij}\})$ and $S_P(\{\xi\})$ $S_Y(\{\gamma_{ij}\})$ on extremal vectors of the $SU(2)_p$ irreps with fixed values of p determine different classes of polarization squeezed states characterized by values of the total polarization noises $(\Delta P)^2$ (determining a "hardness" of squeezing) and of partial coefficients $k_i^P = (\Delta P_i)^2/(\Delta P)^2$ (indicators of polarization squeezing in a given class of GCS).

Furthermore, maximal suppressions of the total polarization noises are attained for states corresponding to quantum unpolarized light. However, actions of operators $S_P(\{\xi\})$ on Glauber's CS (3.14) (with $\alpha_i^\pm \neq 0$), including states of coherent unpolarized light characterized by special values of parameters $\alpha_i^\pm$:

$$\sum_{j=1}^{m}\{[|\alpha_j^+|^2 - |\alpha_j^-|^2] = 0, \quad \sum_{j=1}^{m}[\alpha_j^- \alpha_j^{+*}] = 0, \tag{4.7}$$

do not lead to GCS displaying any polarization squeezing, although GCS (3.15) have a bigger degree of "polarization quasiclassicality" in comparison with GCS (3.12)-(3.13) (cf. Eqs. (4.2a) and (4.6a)).

This situation is similar to that for thermal unpolarized light as is seen from counterparts of Eqs. (4.2), (4.7) for states with $\rho_{th}(1)$ of Eq. (3.20) [23]:

$$a)(\Delta P)^2 = \frac{3}{2}\exp(-\beta)[1-\exp(-\beta)]^{-2} = \frac{3}{4}(< N > + < N >^2/2),$$

$$(\Delta P_\alpha)^2 = \frac{1}{4}(< N > + < N >^2/2), \ \alpha = 1, 2, 0 \tag{4.8a}$$

$$b) |< |P_\gamma| >| = 0, \ \gamma = 0, 1, 2 \tag{4.8b}$$

At the same time, a phase randomization of GCS (3.14) satisfying the condition $|\alpha_i^+| = |\alpha_i^-|$, leads to an intermediate type of unpolarized light revealing features of a "soft" squeezing (but due to another mechanism in comparison with quantum correlations for GCS (3.10)- (3.13)). Specifically, in the simplest case of one ST mode, described by the density matrix

$$\rho_{c/r} = (2\pi)^{-1}\int d\chi |\{\alpha_1^\pm\} >< \{\alpha_1^\pm\}|, \alpha_1^+ = \alpha_1^- exp(i\chi), \tag{4.9}$$

analogs of Eqs. (4.2), (4.8) take the form [23]

$$a)(\Delta P)^2 = \frac{1}{4}(3 < N > + < N >^2), \quad (\Delta P_0)^2 = \frac{1}{4} < N >,$$

$$(\Delta P_\alpha)^2 = \frac{1}{4}(< N > + < N >^2/2), \ \alpha = 1, 2, \tag{4.10a}$$

$$b) |< |P_\gamma| >| = 0, \ \gamma = 0, 1, 2 \tag{4.10b}$$

All this leads to a new classification of states of unpolarized light within quantum optics[10]; herewith different classes of unpolarized light are distinguished by an availability (or not) of polarization (and biphoton) squeezing and its "hardness" as well as by values of the quantitative characteristics of light depolarization $dep_P = (1 - 2\bar{P}/\bar{N})$ and $dep_{P_0} = (1 - |2\bar{\pi}|/\bar{N})$ yielded by the P-spin formalism [2,10].

A similar (but characterized additionally by values of the polarization degree $degP$) classification can be obtained for partially polarized light as it follows from the analysis above for GCS with $p \neq 0$. However, in real physical experimental situations states of light beams do not belong to a single subspace $L(p\mu)$ but are superpositions of states from different subspaces $L(p\mu)$. Therefore, it is of interest to study polarization squeezing properties of partially polarized light beams obtained by actions of the biphoton squeezed operators S_Y, S_X together with the polarization squeezed operators $S_P(\xi)$ on states $|in>$ of some input light beams. As a result we can

obtain new classes of non-classical states of partially polarized light. Without dwelling here on an analysis of this topic we note that due to Eq. (2.8) actions of operators S_X do not change $(\Delta P_\alpha)^2$ but decrease $(\delta P)^2$ in comparison with those of input beams; however, it is not the case for actions of operators S_Y.

5 Geometric phases of polarizaton coherent states

Another area of applications of results obtained in Section 3 is calculations of Pancharatnam-type geometric phases acquired by polarization GCS during the cyclic evolution on the Poincaré sphere. In classical polarization optics it is well known [25,35], that during the cyclic evolution of its polarization state the classical plane wave acquires an additional phase shift equal to half the solid angle subtended by the trajectory of the tip of the Stokes vector on the Poincaré sphere. This additional phase is shown to be a particular case of the Pancharatnam's phase [36] associated with the $SU(2)$ symmetry of the polarization states. It is invariant with respect to deformations of the trajectory leaving the solid angle unchanged and, therefore, is of purely geometric nature. Pancharatnam's ideas have been used [18] to set a generalized definition of the geometric phase, valid for a wide class of quantum evolutions, generally, non-cyclic.

A natural question arises, what happens to the states of quantum light in the similar situation. The considerations presented above make it possible to apply the general definitions [18] of the geometric phase γ to the polarization GCS, since the angles θ, φ enter the appropriate expressions explicitly as classical parameters[19]:

$$\gamma = -\oint_C A_s ds, \tag{5.1}$$

where the gauge potential A_s is expressed as

$$A_s = Im\langle \xi(\theta,\varphi),\psi_0|\frac{d}{ds}|\xi(\theta,\varphi),\psi_0\rangle = Im\langle \xi(\theta,\varphi),\psi_0|\nabla_{\vec{\Omega}}|\xi(\theta,\varphi),\psi_0\rangle\frac{d\vec{\Omega}}{ds}, \tag{5.2}$$

s is an evolution variable which determines the motion of the system along the evolution trajectory C. The states $|\xi(\theta,\varphi),\psi_0\rangle$ are supposed to be the normalized polarization GCS defined by Eq. (3.1) with a certain particular choice of the reference state vectors mentioned above. Then, using explicit forms of the derivatives on the unit sphere,

$$a)\frac{d\vec{\Omega}}{ds} = \vec{e}_\theta\frac{d\theta}{ds} + \vec{e}_\varphi\sin\theta\frac{d\varphi}{ds}, \tag{5.3a}$$

$$b)\langle \xi(\theta,\varphi),\psi_0|\nabla_{\vec{\Omega}}|\xi(\theta,\varphi),\psi_0\rangle = [\vec{e}_\theta\frac{\partial}{\partial\theta'} + \frac{\vec{e}_\varphi}{\sin\theta}\frac{\partial}{\partial\varphi'}]\langle \xi(\theta,\varphi),\psi_0|\xi(\theta',\varphi'),\psi_0\rangle|_{\theta'=\theta,\varphi'=\varphi}, \tag{5.3b}$$

and expressions for GCS overlap integrals one can calculate appropriaite geometric phases (5.1).

As the first example let us consider the general semi-coherent polarization GCS $|\theta,\varphi;p,\mu,n,\lambda\rangle$. Then, substituting the expression (3.5) for the overlap integral in Eq. (5.3b), calculating derivatives and using the definition (5.1)-(5.2) we find

$$\gamma = -2\mu\oint_C \sin^2\frac{\theta}{2}d\varphi, \tag{5.4}$$

which is the -2μ multiple to half the solid angle subtended by C on the Poincaré sphere. In particular, for $\mu = 1/2$ this result coincides with that for classical plane waves, and for $\mu = 0$ we get the complete absence of any geometric phase. A similar result can be found for other polarization GCS associated with the decomposition (2.5) of the Fock space. For example, for the GCS (3.10), (3.13) one gets

$$\gamma = \mp 2p \oint_C \sin^2 \frac{\theta}{2} d\varphi, \tag{5.5}$$

where $p = n/2$ for $m = 1$ and $0 \leq p \leq n/2$ for $m \geq 2$ whereas for GCS (3.8*) the factor $\mp 2p$ is replaced by $\sum_j (n_j^+ - n_j^-)$.

However, in the case of the GCS (3.15), using standard expressions (like Eqs. (3.23b)) [11,13] for their overlap integrals, one gets a more complicated expression

$$\gamma = \gamma^{(0)} + \gamma^{(1)} + \gamma^{(2)} \tag{5.6}$$

where

$$\gamma^{(0)} = 2 < P_0 > \oint_C \sin^2 \frac{\theta}{2} d\varphi, \quad 2 < P_0 >= \sum_{j=1}^{m} \left(|\alpha_j^+|^2 - |\alpha_j^-|^2 \right), \tag{5.7a}$$

$$\gamma^{(1)} = - < P_1 > \oint_C [\sin\theta \cos\varphi d\varphi + \sin\varphi d\theta], \quad < P_1 >= Re[\sum_{j=1}^{m} \alpha_j^- (\alpha_j^+)^*], \tag{5.7b}$$

$$\gamma^{(2)} =< P_2 > \oint_C [\sin\theta \sin\varphi d\varphi - \cos\varphi d\theta], \quad < P_2 >= -Im[\sum_{j=1}^{m} \alpha_j^- (\alpha_j^+)^*] \tag{5.7c}$$

As is seen from Eqs. (5.1), (5.6), (5.7), the total geometric phase for Glauber's GCS is the sum of single ST mode contributions (because of the additivity of the P-quasispin components) that can be used for measurements of the total geometric phases with the help of single-mode interferometric schemes [26,27]. Besides, the contribution of $\gamma^{(0)}$ to the total geometric phase (5.6) is just the classical half the solid angle subtended by the cyclic evolution loop C on the Poincaré sphere, multiplied by the factor $2 < P_0 >$ characterizing polarization structure of the field. If $\sum_{j=1}^m \alpha_j^- (\alpha_j^+)^* = 0$ (that occurs, e.g., in the case when for each j either α_j^+ or α_j^- equals zero) then $\gamma^{(1)}$and $\gamma^{(2)}$ vanish, and Eq.(5.7a) represents the total geometric phase. This is valid, in particular, for the single-mode states (3.16) associated to elliptically polarized waves and obtained by means of transmissions of coherent light beams with a definite circular polarization through polarization rotators. However, it is not the case for general GCS (3.15) with reference vectors (3.14) constrained by conditions (3.17); specifically, even in the case of one ST mode $\gamma^{(a)}, a = 1, 2$ do not vanish that reflects a specific correlation of polarization modes after a transmission of beams (3.14), with $\alpha_j^\pm$ being purely real, through "polarization rotators" (3.2). In general, Eqs. (5.6)-(5.7) describe a structure (nature) of influences of polarization rotators on initial Glauber's CS in dependence on their polarization properties since "energetic" multipliers in these equations are related to expectation values $< P_\alpha >$ of different components of the polarization quasispin P_α. Furthermore, as it follows from Eqs. (5.4)-(5.7), pure quantum states of unpolarized light do not acquire any geometric phases that can be used as an indicator of these states.

6 Conclusion

So, we have defined and examined in both mathematical and physical aspects different types of polarization GCS that enabled to give a quasiclassical description of polarization structure of quantum light beams and to determine different sorts of squeezing in polarization quantum optics. On this way a new classification of polarization states of light beams was given. The polarization GCS of $SU(2)_p$ group obtained above may be also applied as many-parametric test wave functions to analyze other aspects of the polarization quasiclassical description. Among these one should mention applications in "hidden variable" theories [21,37] and studies of other types of the polarization, biphoton and quadrature uncertainty relations and squeezing (cf. [32-34]). Specifically, GCS (3.11) manifest [34] a squeezing for quadrature components of multimode fields determined according to Ref. [8]. Besides, the Y-biphoton squeezed operators $S_Y(\{\gamma_{ij}\})$ represent a particular case of multimode squeezed operators introduced in [9] whereas the X-biphoton squeezed operators $S_X(\{\zeta_{ij}\}$ are their skew-symmetric analogs. It is also of interest to consider generalizations of such GCS obtained when replacing operators $S_P(\{\xi\})$, $S_X(\zeta_{ij}\})$ and $S_Y(\{\gamma_{ij}\})$ by their "quantum" analogs associated with nonlinear, purely quantum versions of Hamiltonians (2.10), (3.2) and polynomial Lie algebras [38].

We have also calculated the occurence of geometric phase in different polarization GCS due to the cyclic evolution in the space of the angular coordinates θ, φ of the "classical" P-quasispin on the Poincaré sphere. The explicit expressions of the geometric phase are shown to depend on the polarization structure of the reference state vectors characterized by expectation values of the P- quasispin components that may be useful for a practical identification of these states. The expressions derived can be used in further investigations of the geometric phases in quantum optics, e.g., by means of using expansions of arbitrary pure quantum states of light in series of orthonormalized states $|p, \mu; n, \lambda >$ and Eq. (5.4).

In conclusion we note that, as whole, results obtained together with those of Part I give a new insight into polarization stucture of multimode quantum light beams that opens a prospect for both applications and further investigations, in particular, search of simplest ways to produce new polarization states of light, studies of interaction of light in these states with material media (including optically active molecules and atoms [34,39-41]) and its propagation through different optical devices as well as the general problem of the description of the quantum light field phase [24]. Notice, however, that in practice it is easier to realize GCS corresponding to Eqs. (3.12) (for $\kappa = 0$) and (3.13) rather than Eqs. (3.10)-(3.11) because the latter require either parametric oscillator crystals with highly anisotropic properties [10] or a synchronous use of two parametric generators with pumping by one laser source [23]. Therefore, perhaps, for production of P-scalar light it is preferable to combine more simple schemes of P_0-scalar light generation together with some interferometric schemes [10]. It is also of interest to consider different schemes related to cascade processes (cf. [42]).

Furthermore, squeezing properties described by Eqs. (4.3)-(4.6) and their analogs can be used for designing different experiments related to the EPR-paradox and entangled states [20-22] and for precise measurements in spectroscopy of anisotropic media [23]. Specifically, properties given by Eqs. (4.5) may be interpreted as an "inversion" of the classical EPR-paradox [43]. It is also of interest to determine the orbital angular momentum for new polarization states as a characterization of their spatial properties (cf. [44]). Besides, the analysis above can be generalized

for arbitrary boson-fermion systems with any $SU(n), n \geq 2$ internal symmetries (cf. [1,15,45]).

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